GROUP PROPERTIES AND INVARIANT SOLUTIONS OF EQUATIONS OF AN ELECTRIC FIELD IN THE CASE OF NONLINEAR OHM'S LAW

Yu. P. Emets and V. B. Taranov

UDC 538.4

The group properties of one-dimensional nonstationary equations of an electric field in homogeneous isotropic media with nonlinear conductivity are considered. The nonlinear Ohm's laws for which these equations have the broadest symmetry properties are determined. Ordinary differential equations determining invariants solutions are obtained; the order of the equations is lowered or they are integrated to the end.

Deviations from the linear Ohm's law are observed already in comparatively weak fields for such conducting media as plasma and semiconductors. In calculating the electromagnetic fields, Ohm's law is usually taken from experimental data or calculated theoretically within the framework of a concrete physical model of conductivity.

Here, the problem is posed differently: an arbitrary form of Ohm's law $j = j(E)$ is taken and one-dimensional nonstationary equations of an electric field in a homogeneous medium with consideration of displacement currents (nonlinear telegraphic equation), and in a quasistationary approximation (nonlinear equations of heat conduction) are considered. In conducting a group classification of these equations we isolate a set of Ohm's laws for which symmetry of the equations is broadened. It turns out that these Ohm's laws are the same for both equations and the group admissible by the nonlinear telegraphic equation is the subgroup of the group admissible by the nonlinear equation of heat conduction.

It is interesting that the dependences $j(E)$ obtained thereby are realized in practice in a sufficiently wide range of variations of j and E and correspond to the nonlinear Ohm's laws found in plasma and semiconductors.

The group classification of the nonlinear equation of heat conduction is given by L. V. Ovsyannikov in [1], where the form of all invariant solutions was also obtained. Here, we will give a parallel group classification of both nonlinear equations – telegraphic and heat conduction (the latter is taken in the form of the equation for the electric field strength E more usual for electrodynamics) – and we find the explicit form of the invariant solutions or the simplest form of ordinary differential equations which determine the invariant solutions.

1. For the one-dimensional model

$$\mathbf{E} = \{0, 0, E(x, t)\}, \qquad \mathbf{H} = \{0, H(x, t), 0\}$$

the Maxwell equations

$$\frac{\partial E}{\partial x} = \frac{\mu}{c}\frac{\partial H}{\partial t}, \qquad \frac{\partial H}{\partial x} = \frac{\varepsilon}{c}\frac{\partial E}{\partial t} + \frac{4\pi}{c} j(E) \tag{1.1}$$

Kiev. Translated from Zhurnal Prikladnoi Mekhanika i Tekhnicheskaya Fizika, No. 3, pp. 28–36, May-June, 1972. Original article submitted June 14, 1971.

lead to the nonlinear telegraphic equation for E

$$\frac{\partial^2 E}{\partial x^2}-\frac{\partial^2 E}{\partial t^2}-\sigma(E)\frac{\partial E}{\partial t}=0, \qquad \sigma(E)\equiv\frac{dj}{dE} \tag{1.2}$$

where the transformation

$$x^*=\frac{4\pi}{c}\sqrt{\frac{\mu}{\varepsilon}}\,x, \qquad t^*=\frac{4\pi}{\varepsilon}t$$

is assumed fulfilled, and the asterisks are dropped.

For quasi-stationary processes, when we can neglect displacement currents, we arrive at the equation

$$\frac{\partial^2 E}{\partial x^2}-\sigma(E)\frac{\partial E}{\partial t}=0 \tag{1.3}$$

If we write this equation in symbols of current density j, it coincides in form with the equation analyzed by L. V. Lvsyannikov in [1]. Actually, transforming Ohm's law, we obtain

$$\frac{\partial}{\partial x}\left[\rho(j)\frac{\partial j}{\partial x}\right]=\frac{\partial j}{\partial t}, \qquad \rho(j)\equiv\frac{dE(j)}{dj} \tag{1.4}$$

Hence, we see that j corresponds to the temperature and the differential resistance $\rho(j)$ to the coefficient of heat conduction of the substance.

It should be noted that the problem of the criterion of quasi-stationarity of the process in nonlinear electrodynamics is complicated considerably and its examination is beyond the scope of this article.

2. In determining the continuous group of transformations G_r, admissible by Eq. (1.2), we proceed, according to the general theory [2-4], from the condition that (1.2) determines the relative invariant of a twice continuous group $G_r^{(2)}$.

For the vectors $\xi^i(x, t, E)$ $(i = 1, 2, 3)$ of the one-parameter group

$$X=e^a X_a=\xi^1\frac{\partial}{\partial x}+\xi^2\frac{\partial}{\partial t}+\xi^3\frac{\partial}{\partial E}$$
$$(\xi^i=e^k\xi_k^i,\ e^k=\text{const},\ k=1,\ldots,r) \tag{2.1}$$

we obtain a system of characteristic equations of infinitesimal transformations

$$\begin{gathered}\xi^3_{xx}-\xi_{tt}^3-\sigma\xi_t^3=0,\quad 2\xi^3_{tE}+\sigma\xi_t^2+\sigma'\xi^3=0\\ 2\xi^3_{xE}+\sigma\xi_x^2=0,\quad \xi_x^1-\xi_t^2=0,\quad \xi_t^1-\xi_x^2=0\\ \xi_E^1=\xi_E^2=\xi^3_{EE}=0,\quad \sigma\equiv dj/dE\end{gathered} \tag{2.2}$$

Here, the lower indices denote differentiation with respect to the corresponding variables, and the prime denotes the complete derivative with respect to E.

In the case of nonlinear conduction ($\sigma'(E)\neq 0$) the solution of system (2.2) has the form

$$\xi^1=e^1x+e^2,\quad \xi^2=e^1t+e^3,\quad \xi^3=e^4E+e^5 \tag{2.3}$$

whereby the dependence $\sigma(E)$ obeys the condition

$$(e^4E+e^5)\,\sigma'+e^1\sigma=0 \tag{2.4}$$

In the most general case, when Ohm's law is arbitrary (and consequently, the differential conduction $\sigma(E)$ can be arbitrary), fulfillment of (2.4) is possible only when

$$e^1=e^4=e^5=0$$

and the base of the group is formed by the operators

$$X_1=\frac{\partial}{\partial x}, \qquad X_2=\frac{\partial}{\partial t} \tag{2.5}$$

Broadening of the group is possible on limiting the arbitrariness in selecting the function $\sigma(E)$ by assuming that it satisfies condition (2.4) with nonzero constants.

In this case, it follows from the condition $\sigma'(E) \neq 0$ that only one of the constants in (2.4) can be selected as arbitrary. Without limiting generality, we can assume that this is e^1. Then, $e^4 = a e^1$, $e^5 = b e^1$, where a and b are fixed constants.

In this case, the group is broadened by the addition of the operator

$$X_3 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + (aE + b)\frac{\partial}{\partial E} \tag{2.6}$$

Integrating (2.4), we obtain

$$\sigma(E) = \begin{cases} c\,(aE + b)^{-1/a} & \text{when } a \neq 0 \\ c\exp(-E/b) & \text{when } a = 0 \end{cases} \tag{2.7}$$

and then, taking into account that $\sigma(E) = dj/dE$, we find

$$j(E) = \begin{cases} \frac{c}{a-1}(aE + b)^{1-1/a} + d & \text{when } a \neq 0,\ a \neq 1 \\ c\ln(E + b) + d & \text{when } a = 1 \\ -bc\exp(-E/b) + d & \text{when } a = 0 \end{cases} \tag{2.8}$$

The sense of constants a, b, c, and d in Eqs. (2.7), (2.8) is determined by the physical content of the problems being considered. If, for example, we set $j = 0$ when $E = 0$, which corresponds to the assumption of the absence of extraneous currents in the medium, then, Eqs. (2.8) take the form

$$\frac{j(E)}{j(E_0)} = 1 - \left(1 - \frac{E}{E_0}\right)^{\alpha} \quad \text{when } a > 1 \text{ or } a < 0 \quad (\alpha \equiv 1 - 1/a > 0) \tag{2.9}$$

$$\frac{j(E)}{j(\infty)} = 1 - \left(1 + \frac{E}{E_0}\right)^{\alpha} \quad \text{when } 0 < a < 1 \quad (\alpha < 0) \tag{2.10}$$

$$\frac{j(E)}{j_0} = \ln\left(1 + \frac{E}{E_0}\right) \quad \text{when } a = 1 \quad (j_0 \equiv j((e-1)E_0)) \tag{2.11}$$

$$\frac{j(E)}{j(\infty)} = 1 - \exp(-E/E_0) \quad \text{when } a = 0 \tag{2.12}$$

Such dependences, or more exactly some of their branches, are realized in practice in a sufficiently wide range of variation of j and E. Thus, dependence (2.12) is the nonlinear Ohm's law of high-temperature plasma in the case of the development of an acoustic instability in it [5]. In low-temperature plasma with nonequilibrium conductivity Ohm's law has the form of (2.9) [6].

3. The group classification of the equation of the field for quasi-stationary processes (1.3) is carried out in the same manner as the preceding case. To find the vector ξ^i, we obtain a system of characteristic equations

$$\begin{gathered} \xi^3_{xx} - \sigma\xi_t^3 = 0, \quad \xi^1_{xx} - 2\xi^3_{xE} - \sigma\xi_t^1 = 0 \\ (2\xi_x^1 - \xi_t^2)\sigma + \xi^3\sigma' = 0, \quad \xi_E^1 = \xi_E^2 = \xi_x^2 = \xi^3_{EE} = 0 \end{gathered} \tag{3.1}$$

An analysis of the solutions of this system leads to the following result: for an arbitrary law of nonlinearity of the medium, the base of the group admissible by Eq. (1.3) is formed by the operators

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} \tag{3.2}$$

Broadening of the group occurs for the same Ohm's laws (2.8), which leads to broadening of the group of the nonlinear telegraphic equation. In this case, the operator

$$X_4 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + (aE + b)\frac{\partial}{\partial E} \tag{3.3}$$

is added.

Further broadening of the group occurs for the same particular value $a = 1/4$, and there is an additional operator

$$X_5 = x^2\frac{\partial}{\partial x} + x(E + 4b)\frac{\partial}{\partial E} \tag{3.4}$$

As we see, the Lie group of the nonlinear telegraphic equation (1.2) is a subgroup of the group admissible by the nonlinear equation of heat conduction (1.3).

4. Proceeding to finding the invariant solutions of rank 1 of Eq. (1.2), we make certain simplifications. For an arbitrary Ohm's law this equation admits a group with operators

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial t} \tag{4.1}$$

Broadening of the group occurs when

$$\sigma(E) = \begin{cases} \exp(-E) & \text{(A)} \\ E^{-1/a} & \text{(B)} \end{cases} \tag{4.2}$$

The entire diversity (2.7) of nonlinear functions $\sigma(E)$ reduces to these two main cases. If (4.2) occurs, the operator

$$X_3 = \begin{cases} x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \frac{\partial}{\partial E} & \text{(A)} \quad (4.3) \\ x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + aE\frac{\partial}{\partial E} & \text{(B)} \quad (4.4) \end{cases}$$

is added.

To obtain essentially different invariant solutions, we must construct an optimal system of dissimilar subgroups [1]. In the case of the telegraphic equation the number of operators is small, and we can limit ourselves to simple inspection of the one-parameter subgroups.

1°. We will consider the solution on a one-parameter subgroup $X = X_1 + \alpha^{-1}X_2$ admissible by Eq. (1.2) for arbitrary $\sigma(E)$.

The invariants of the operator X determine the form of the solutions (traveling waves) $E = E(x - \alpha t)$, for which follows from (1.2) the equation:

$$(1 - \alpha^2)\frac{d^2F}{d\xi^2} + \alpha\sigma(E)\frac{dE}{d\xi} = 0 \qquad (\xi = x - \alpha t) \tag{4.5}$$

Hence, we see that, just as for the linear telegraphic equation (σ = const), waves propagating with velocity $\alpha = \pm 1$ do not exist. For all waves with velocities $\alpha \neq \pm 1$ and $\alpha = 0$, the transformation $\xi^* = \alpha\xi/(1-\alpha^2)$ reduces Eq. (4.5) to the form (we drop the asterisks)

$$\frac{d^2E}{d\xi^2} + \sigma(E)\frac{dE}{d\xi} = 0 \tag{4.6}$$

In case (A) (4.6) has the solution

$$E = \begin{cases} \ln\{(1/C_1)[\exp(C_1\xi + C_2) - 1]\}, & C_1 \neq 0 \\ \ln(\xi + C_2) \end{cases} \tag{4.7}$$

where C_1 and C_2 are constants of integration.

In case (B) the solution is given by the formulas

$$\xi + C_2 = \int \frac{dE}{C_1 - \ln E} \qquad \text{when } a = 1$$

$$E = \left(\frac{m+1}{m}\xi + C_2\right)^{1/(m+1)} \qquad \text{when } a \neq 1,\ a \neq 0$$
$$C_1\xi + C_2 = E - \frac{1}{C_1 m}\int \frac{dE}{E^m + 1/C_1 m} \qquad m \equiv \frac{1}{a} - 1 \tag{4.8}$$

The last solution can be rewritten in a form convenient for analysis

$$\xi = E \mp \int_0^E \frac{du}{u^m \pm 1}, \qquad m \neq 0 \tag{4.9}$$

assuming (depending on the value of m)

$$1 / C_1 m = \pm q^m \qquad (q > 0)$$

performing the transformation

$$E = qE^*, \quad C_1\xi + C_2 = q\xi^*$$

and dropping the asterisks. We will write out these solutions in the case of the plus sign in (4.9) for certain values of m

$$\xi = E - \ln(1 + E) \qquad \text{when } m = 1 \tag{4.10}$$

$$\xi = E - \operatorname{arc\,tg} E \qquad \text{when } m = 2 \tag{4.11}$$

$$\xi = E - \frac{\pi}{6\sqrt{3}} - \frac{1}{6}\ln\left[\frac{(E+1)^2}{E^2 - E + 1}\right] - \frac{1}{\sqrt{3}}\operatorname{arc\,tg}\left(\frac{2E-1}{\sqrt{3}}\right) \tag{4.12}$$

when m = 3 ($a = 1/4$).

Thus, the nonlinear telegraphic equation allows a solution of the traveling-wave type of different form, unlike the linear which allows waves only of a special form (exponential).

2°. We will consider invariant solutions on subgroup X_3. The solutions have the form

$$E = \begin{cases} \ln[tF(\xi)] & \text{(A)} \\ [tF(\xi)]^a & \text{(B)} \end{cases} \tag{4.13}$$

where $\xi = x/t$.

Equation (1.2) leads to an ordinary differential equation for $F(\xi)$

$$\left[(1-\xi^2)\frac{F'}{F} + \xi - \frac{\xi}{F}\right]' + a\left[(1-\xi)\frac{F'}{F} + 1\right]\left[(1+\xi)\frac{F'}{F} - 1\right] = 0 \tag{4.14}$$

The value $a = 0$, for which Eq. (4.14) is easily integrated, corresponds to case (A)

$$E = \begin{cases} \ln\left[\frac{t + C_1 x}{1 - C_1^2} + C_2\sqrt{|t^2 - x^2|}\left|\frac{x+t}{x-t}\right|^{C_1/2}\right], & C_1^2 \neq 1 \\ \ln\left[\mp\frac{x}{2} + C_2(x \pm t) + \frac{1}{4}(x \pm t)\ln\left|\frac{x+t}{x-t}\right|\right] \end{cases} \tag{4.15}$$

5. We will now consider the invariant solutions of rank 1 of the nonlinear equation of heat conduction (1.3) which allows for an arbitrary $\sigma(E)$ the group

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} \tag{5.1}$$

The group is broadened in the case of special functions $\sigma(E)$ given in (4.2)

$$X_4 = \begin{cases} x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \frac{\partial}{\partial E} & \text{(A)} \\ x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + aE\frac{\partial}{\partial E} & \text{(B)} \end{cases} \tag{5.2}$$

In addition, in case (B) when $a = 1/4$, the operator

$$X_5 = x^2 \frac{\partial}{\partial x} + xE \frac{\partial}{\partial E} \tag{5.3}$$

is added.

When obtaining the invariant solutions, we will be guided by the optimal system of dissimilar subgroups found by L. V. Ovsyannikov [1].

1°. Solutions of the wave type $E = E(x - \alpha t)$ are determined now from the equation

$$\frac{d^2E}{d\xi^2} + \alpha\sigma(E) \frac{dE}{d\xi} = 0, \qquad \xi = x - \alpha t \tag{5.4}$$

Unlike the preceding case (4.5), the waves propagate with arbitrary velocities. Performing the transformation $\xi^* = \alpha\xi$ (the asterisks will be omitted below), we reduce Eq. (5.4) to the form of (4.6), which permits extending the results of paragraph 1° of the preceding section to the case under consideration.

2°. We will consider the solution on subgroup $X = \alpha X_3 + X_4$, having the form

$$E = \ln [x^{1/1+\alpha} e^{F(\xi)}], \quad E = [x^{1/1+\alpha} e^{F(\xi)}]^a$$

in cases (A) and (B), respectively. Here, $\xi = xt^{-(1+\alpha)/(1+2\alpha)}$, and F satisfies the equation

$$\xi^2 F'' + \frac{1+\alpha}{1+2\alpha} \xi^{(1+2\alpha)/(1+\alpha)} e^{-F} \xi F' + a\left(\frac{1}{1+\alpha} + \xi F'\right)^2 - \frac{1}{1+\alpha} = 0 \tag{5.5}$$

$a = 0$ corresponding to case (A). By direct check, we can be convinced that Eq. (5.5) admits the operator

$$Y = \xi \frac{\partial}{\partial \xi} + \frac{1+2\alpha}{1+\alpha} \frac{\partial}{\partial F} \tag{5.6}$$

By means of Y, we obtain

$$\xi = e^z, \qquad F = \Phi(z) + \frac{1+2\alpha}{1+\alpha} z \tag{5.7}$$

$$\Phi'' - \left(1 - \frac{1+\alpha}{1+2\alpha} e^{-\Phi}\right)\Phi' - 2\left(1 - \frac{1}{2} e^{-\Phi}\right) + a(\Phi' + 2)^2 = 0 \tag{5.8}$$

In case (A) for $\alpha = 0$ ($X = X_4$) Eq. (5.8) is integrated to the end and leads to the following expressions for E:

$$E = \begin{cases} \ln \left[C_2 t \exp\left(C_1 \frac{x}{t}\right) - \frac{1}{C_1}\left(x + \frac{t}{C_1}\right)\right], & C_1 \neq 0 \\ \ln [C_2 t + x^2 / 2t] \end{cases} \tag{5.9}$$

For the other limiting value $\alpha \to \infty$ (self-preserving solution on operator X_3) Eq. (5.8) admits the first integral

$$\Phi' - 2 \ln (\Phi' + 2) - {}^1/_2\, e^{-\Phi} - \Phi = C_1 \tag{5.10}$$

In case (B), there is a particular solution of the form

$$E = \left[\frac{x^2}{2(1-2a)t}\right]^a, \qquad a \neq \frac{1}{2} \tag{5.11}$$

3°. The solution on subgroup $X = \alpha^{-1} X_1 - X_3 + X_4$. Here, we have, respectively, for (A) and (B)

$$E = \ln\left[\frac{1}{t} e^{F(\xi)}\right], \quad E = \left[\frac{1}{t} e^{F(\xi)}\right]^a \quad (\xi = te^{\alpha x})$$

and F is determined from the equation

$$\alpha^2\xi^2 F'' + \alpha^2 \xi F' + e^{-F}(1 - \xi F') + a\alpha^2\xi^2 (F')^2 = 0 \tag{5.12}$$

If $a = 0$, (5.12) admits the first integral ($\xi = e^{z}$)

$$\frac{dF}{dz} + \ln\left(\frac{dF}{dz} - 1\right) + \frac{1}{\alpha^2} e^{-F} = C_1 \tag{5.13}$$

4°. We proceed to the subgroup $\alpha X_2 - {}^1/_2 X_3 + X_4$. For (A) and (B), we have

$$E = \ln [x^2 e^{F(\xi)}], \qquad E = [x^2 e^{F(\xi)}]^a$$

where $\xi = e^t x^{-2\alpha}$ and F satisfies

$$2\alpha^2\xi^2 F'' + 2\alpha^2\xi F' + \alpha\xi F' - 1 - {}^1/_2 e^{-F}\xi F' + 2a(1 - \alpha\xi F')^2 = 0 \tag{5.14}$$

whence, follows:

$$2\frac{d^2F}{dy^2} + \frac{dF}{dy} - \frac{1}{2\alpha} e^{-F} \frac{dF}{dy} + 2a\left(1 - \frac{dF}{dy}\right)^2 - 1 = 0 \tag{5.15}$$

where $y = \alpha^{-1} \ln \xi$. When $a = 0$, Eq. (5.15) admits the first integral

$$2\frac{dF}{dy} + \frac{1}{2\alpha} e^{-F} + F - y = C_1 \tag{5.16}$$

5°. In case (B), when $a = {}^1\!/_4$, the nonlinear equation of heat conduction is integrated to the end on subgroup $\alpha X_2 + X_5$. The solutions have the form

$$E = xF(\xi), \quad \xi = t + \alpha/x$$

For F, we obtain the equation

$$\alpha^2 \frac{d^2F}{d\xi^2} - \frac{1}{F^4} \frac{dF}{d\xi} = 0 \tag{5.17}$$

integrating which, we have the expression for E

$$E = x\left[-\frac{4}{3\alpha^2}\left(C_2 + t + \frac{\alpha}{x}\right)\right]^{1/4}$$

$$C_1\left(t + \frac{\alpha}{x}\right) + C_2 = 3\alpha^2 C_1 \frac{E}{x} + \frac{1}{6}\ln\left\{\left(a - \frac{E}{x}\right)^2\left[a^2 + a\frac{E}{x} + \left(\frac{E}{x}\right)^2\right]^{-1}\right\} -$$
$$- \frac{1}{a^2\sqrt{3}} \operatorname{arc\,tg}\left[a^{-1}3^{-1/2}\left(2\frac{E}{x} + a\right)\right] \quad \left(a^3 = \frac{1}{3\alpha^2 C_1}\right) \tag{5.18}$$

6°. We will consider, finally, the solution on subgroup $X_3 + X_5$. Here, we have

$$E = (1 + x)F(\xi), \qquad \xi = \frac{1}{t}\left(\frac{x}{1+x}\right)^2$$

where F obeys the condition

$$\frac{d^2F}{d\xi^2} + \left(\frac{1}{2\xi} + \frac{1}{4F^4}\right)\frac{dF}{d\xi} = 0 \tag{5.19}$$

which by means of substitution

$$\xi = e^{4z}, \quad F = e^z \Phi(z)$$

is reduced to the form

$$P\left(\frac{dP}{d\Phi} + \frac{1}{\Phi^4}\right) - \Phi\left(1 - \frac{1}{\Phi^4}\right) = 0, \qquad P(\Phi) = \frac{d\Phi}{dz} \tag{5.20}$$

Equation (5.20) admits the particular solution $\Phi = 1$, to which corresponds

$$E = [x(1 + x)]^{1/2} t^{-1/4} \tag{5.21}$$

7°. We will consider the solution on subgroup X_3 admissible by Eq. (1.3) for arbitrary $\sigma(E)$. The invariants of operator X_3

$$I_1 = x^2 / 4t, \quad I_2 = E$$

determine the form of the unknown solutions

$$E = E(\xi), \ \xi = x^2 / 4t$$

and from (1.3) follows the equation

$$\frac{d^2E}{d\xi^2} + \left[\frac{1}{2\xi} + \sigma(E)\right]\frac{dE}{d\xi} = 0 \tag{5.22}$$

6. Some of these solutions of the nonlinear equation of heat conduction were obtained earlier by other authors.

The ordinary differential equations for the invariant solutions on subgroup $X_3 + \alpha X_4$ in case (B) were obtained by G. I. Barenblatt [7] by the dimensionality theory method and solved by him approximately.

Equation (5.10) was obtained and solved by T. R. Soldatenkov approximately in an examination of the problem of penetration of an electromagnetic field into plasma (problem of the nonlinear skin effect) [5].

The form of all possible invariant solutions of the nonlinear equation of heat conduction, as already noted, was given by L. V. Ovsyannikov in [1].

LITERATURE CITED

1. L. V. Ovsyannikov, "Group properties of the equation of nonlinear heat conduction," Dokl. Akad. Nauk SSSR, 125, No. 3, 492-495 (1959).
2. N. G. Chebotarev, Lie Group Theory [in Russian], Gostekhteorizdat, Moscow–Leningrad (1940).
3. L. P. Eisenhart, Continuous Transformation Groups [Russian translation], Izd. Inostr. Lit., Moscow (1947).
4. L. V. Ovsyannikov, Group Properties of Differential Equations [in Russian], Izd. SO. AN SSSR, Novosibirsk (1962).
5. T. R. Soldatenkov, "Penetration of an electromagnetic field into a plasma in the case of a nonlinear Ohm's law," Nucl. Fusion, 10, No. 1, 69-73 (1970).
6. J. Kerrebrock, "Electrical conduction of gases at a high electron temperature," in: Engineering Problems in Magnetohydrodynamics [Russian translation], Mir, Moscow (1965).
7. G. I. Barenblatt, "Some unsteady liquid and gas motions in a porous medium," Prikl. Matem. i Mekhan., 16, No. 1, 67-78 (1952).